SOME TESTS FOR COMPARING PERCENTANGE POINTS OF TWO ARBITRARY CONTINUOUS POPULATIONS

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1. Summary. ~~Let us consider~~ samples from two continuous populations, the first with unique $100\alpha\%$ point θ_α, the second with unique $100\beta\%$ point ϕ_β. The two populations are not necessarily the same or even related. This paper presents some easily applied significance tests for $\theta_\alpha - \phi_\beta$ which are approximately valid for moderate and large sized samples. The exact significance level of a test is not known but its value is determined within reasonably close limits. Efficiency properties of these tests are investigated for the special case of normal populations with known ratio of variances. The tests are found to be reasonably efficient if α and β are not too large or too small. Since these tests are often valid for moderate as well as large sized samples, they may be of practical value.

2. Introduction and descriptive outline. A problem of occasional practical interest is that of comparing a specified percentage point θ_α of one arbitrary population with a specified percentage point ϕ_β of another arbitrary population. For example, it might be desired to test whether the 10% point of the first population exceeds the 76% point of the second population by more than 11 units. As another example, one might wish to test whether the 93% point of the first population is the same as the point obtained by subtracting 5 units from the 24% point of the second population. Since little is known concerning the distribution functions of the populations, however, most methods developed for testing $\theta_\alpha - \phi_\beta$ require very large samples. The purpose of this paper is to present some tests of $\theta_\alpha - \phi_\beta$ which are applicable to moderate sized samples for many situations of practical interest.

Before outlining the chain of reasoning used to obtain the results presented in this paper, let us consider a large sample method of obtaining tests for $\theta_\alpha - \phi_\beta$. Both continuous populations are assumed to have probability density functions. Let the first population have a density function $f(x)$ while the second population has a density function $g(y)$. These two functions are arbitrary except that $f(\theta_\alpha) \neq 0$, $g(\phi_\beta) \neq 0$, and $f'(\theta_\alpha)$,

$g'(\phi_\beta)$ exist and are continuous in the vicinity of the specified points. Let $x(1), \cdots, x(m)$ represent the values, arranged in increasing order of magnitude, of a sample of size m from the population with density $f(x)$ while $y(1), \cdots, y(n)$ denote the values, arranged in increasing order of magnitude, of a sample of size n from the population with density $g(y)$. Then asymptotically ($m, n \to \infty$) the distribution of

$$(1) \qquad \left[x(\alpha m) - y(\beta n) - (\theta_\alpha - \phi_\beta)\right] \Big/ \sqrt{\alpha(1-\alpha)/mf(\theta_\alpha)^2 + \beta(1-\beta)/ng(\phi_\beta)^2}$$

is standard normal (i.e., zero mean, unit variance). Here αm and βn are integers. This is a direct application of a modification of the results of [1, p. 369]. For many situations of practical interest, the distribution of (1) is nearly standard normal for values of m and n which are not extremely large (see, e.g., [2]). Thus if $f(\theta_\alpha)$ and $g(\phi_\beta)$ were known, (1) could be used to test $\theta_\alpha - \phi_\beta$ for practical situations involving mediumly large samples. Although $f(\theta_\alpha)$ and $g(\phi_\beta)$ are not known, (1) can be modified to yield large sample tests for $\theta_\alpha - \phi_\beta$. For example, $f(\theta_\alpha)$ and $g(\phi_\beta)$ can be replaced by estimates based on the sample values which converge in probability to $f(\theta_\alpha)$ and $g(\phi_\beta)$ as $m, n \to \infty$. Then asymptotically the distribution of the resulting modification of (1) is standard normal. This follows from combining (1) with the convergence theorem [1, p. 254]. A refinement of this method using the results of [3] could also be applied. Even if $f(\theta_\alpha)$ and $g(\phi_\beta)$ were known, however, use of (1) would not necessarily yield sufficiently accurate results for moderate values of m and n.

This paper develops tests which appear to be reasonably accurate for moderate as well as large values of m and n if the populations are of the type approximated in practical situations. These tests are based on statistics of the form

$$(2) \qquad x(\alpha m + C_1\sqrt{m}) - y(\beta n + C_2\sqrt{n}).$$

By appropriate choice of C_1 and C_2, the values of quantities of the form

$$(3) \qquad \Pr\left[x(\alpha m + C_1\sqrt{m}) - y(\beta n + C_2\sqrt{n}) - (\theta_\alpha - \phi_\beta) < 0\right]$$

can be made to be within fairly small intervals for moderate m and n, where the values within an interval are all of a magnitude suitable for significance levels. Similarly for quantities of the form

$$Pr\left[x(\alpha m + C_1\sqrt{m}) - y(\beta n + C_2\sqrt{n}) - (\theta_\alpha - \phi_\beta) > 0\right].$$

Thus (2) can be used to compare $\theta_\alpha - \phi_\beta$ with a given hypothetical value μ_o. For example, $\theta_\alpha - \phi_\beta < \mu_o$ can be investigated by one-sided tests of the form

(4) $\qquad$ Accept $\theta_\alpha - \phi_\beta < \mu_o$ if $x(\alpha m + C_1\sqrt{m}) - y(\beta n + C_2\sqrt{n}) < \mu_o$.

From (3), the significance level of this test can be fixed within reasonably close limits (for moderate m and n) by suitable choice of C_1 and C_2.

Let us consider an outline of the method used to derive the tests. For this purpose it is sufficient to limit consideration to one-sided tests of the form (4). The first step of the derivation consists in determining the asymptotic distribution of (2) under the assumption that $f(\theta_\alpha)$ and $g(\phi_\beta)$ are known. The result obtained is that the asymptotic distribution of a certain function of (2), $\theta_\alpha - \phi_\beta$, $f(\theta_\alpha)$, $g(\phi_\beta)$, C_1 and C_2 is standard normal. To emphasize the dependence on $f(\theta_\alpha)$ and $g(\phi_\beta)$, let this function by denoted by $Z\left[f(\theta_\alpha), g(\phi_\beta)\right]$.

Since $f(\theta_\alpha)$ and $g(\phi_\beta)$ are not known, it would be convenient if C_1 and C_2 could be chosen so that the value of (3) is independent of the true values of these quantities (asymptotically). This "Studentization" can be accomplished to a reasonable approximation. Let the interval (γ, δ) include the set of possible values of (3) when $f(\theta_\alpha)$ and $g(\phi_\beta)$ are replaced by arbitrary positive numbers. By suitable choice of C_1 and C_2, γ and δ can be made to lie fairly close together and have values suitable for significance levels (asymptotically).

Now consider the case where m and n are not large. Let the function values $f(\theta_\alpha)$ and $g(\phi_\beta)$ be replaced by the arbitrary positive parameters A and B in the function $Z\left[f(\theta_\alpha), g(\phi_\beta)\right]$. By appropriate selection of the values of A and B, the cumulative distribution function (cdf) of $Z[A, B]$ is fitted to the standard normal cdf in the small interval γ to δ. There are intuitive reasons for believing that a reasonably accurate fit can be obtained in this small interval for moderate m and n if the two populations are of the types approximated in practice. Thus for practical situations the significance level of (4) should lie within or near the interval γ to δ for moderate values of m and n.

The basic trick in the derivations lies in selecting C_1 and C_2 so that replacing $f(\theta_\alpha)$ and $g(\phi_\beta)$ by numbers which may vary greatly from the true values of these quantities does not cause much variation in the value of (3). This allows the fitting of the cdf's in the small interval to be performed while keeping the possible significance levels in a fairly narrow range. Since the interval where the fitting takes place is small and a wide latitude in the choice of the parameters is available, it seems reasonable to believe that an accurate fit can be obtained for moderate m and n.

The tests advocated by this paper are stated in section 3. This section also presents some "rule of thumb" conditions for deciding when m and n are sufficiently large for the tests to be applicable.

A detailed derivation of the properties of the tests presented in section 3 is contained in section 4.

To obtain an approximate lower bound for the efficiency of the tests for $\theta_\alpha - \phi_\beta$, the case where both populations are normal and the ratio of variances is known was analyzed for large m and n. The resulting efficiencies should be much lower than those ordinarily encountered because of the additional information assumed and because the efficiency of non-parametric results usually decreases as the sample size increases. It is found that the efficiency of the tests is reasonably high if $.1 \leq \alpha, \beta \leq .9$. Further results and derivations are given in section 5.

To obtain a rough quantitative idea of how large m and n need to be for sufficiently accurate tests, the special case where $\alpha = \beta$ and $f(x) \equiv g(x)$ is considered. Then the significance level of a test is exactly determined and can be obtained without a substantial amount of computation. Section 6 contains an analysis for this case in which the exact results are checked against the range γ to δ. It is found that the true significance level lies within or near the range γ to δ even for fairly small values of m and n.

The procedures used in this paper are based on some methods originally presented by one of the authors in [4].

3. <u>Statement of tests</u>. This section contains explicit specification of the tests discussed in the preceding sections.

First let us consider one-sided tests of $\theta_\alpha - \emptyset_\beta < \mu_o$. Let ϵ be the approximate significance level desired. Then the test advocated is

(5) <u>Accept</u> $\theta_\alpha - \emptyset_\beta < \mu_o$ <u>if</u> $x(\alpha m + .85K_\epsilon\sqrt{\alpha(1-\alpha)m}) - y(\beta n - .85K_\epsilon\sqrt{\beta(1-\beta)n}) < \mu_o$,

where K_ϵ is the standardized normal deviate exceeded with probability ϵ. Here $\alpha m + .85K_\epsilon\sqrt{\alpha(1-\alpha)m}$ and $\beta n - .85K_\epsilon\sqrt{\beta(1-\beta)n}$ should be integers or nearly equal to integers. If $\alpha m + .85K_\epsilon\sqrt{\alpha(1-\alpha)m}$ is not an integer, $x(\alpha m + .85K_\epsilon\sqrt{\alpha(1-\alpha)m})$ has the interpretation x(integer nearest to $\alpha m + .85K_\epsilon\sqrt{\alpha(1-\alpha)m}$). Similarly for $y(\beta n - .85K_\epsilon\sqrt{\beta(1-\beta)n})$. An approximate lower bound for the significance level of (5) is γ, where γ is defined by the relation $K_\gamma = 1.25K_\epsilon$. An approximate upper bound is δ, where $K_\delta = .83K_\epsilon$. For example, let $\epsilon = .05$; then $\gamma = .020$ and $\delta = .086$. As another example, let $\epsilon = .01$; then $\gamma = .0023$ and $\delta = .027$. For most situations of practical interest, it appears likely that the true value of the significance level of (5) will be much nearer γ than δ.

Next consider one-sided tests of $\theta_\alpha - \emptyset_\beta > \mu_o$. If ϵ is the approximate significance level desired, the test advocated is

(6) <u>Accept</u> $\theta_\alpha - \emptyset_\beta > \mu_o$ <u>if</u> $x(\alpha m - .85K_\epsilon\sqrt{\alpha(1-\alpha)m}) - y(\beta n + .85K_\epsilon\sqrt{\beta(1-\beta)n}) > \mu_o$.

Here $x(\alpha m - .85K_\epsilon\sqrt{\alpha(1-\alpha)m})$ and $y(\beta n + .85K_\epsilon\sqrt{\beta(1-\beta)n})$ have interpretations of the type stated for (5) and $\alpha m - K_\epsilon\sqrt{\alpha(1-\alpha)m}$, $\beta n + K_\epsilon\sqrt{\beta(1-\beta)n}$ should be integers or nearly equal to integers. As for test (5), an approximate lower bound for the significance level of (6) is γ while δ is an approximate upper bound.

Two-sided tests of $\theta_\alpha - \emptyset_\beta \neq \mu_o$ can be obtained by combining (5) and (6). For example, a two-sided test with desired significance level approximately 2ϵ and nearly equal tails is given by

<u>Accept</u> $\theta_\alpha - \emptyset_\beta \neq \mu_o$ <u>if either</u>

$$x(\alpha m + .85K_\epsilon\sqrt{\alpha(1-\alpha)m}) - y(\beta n - .85K_\epsilon\sqrt{\beta(1-\beta)n}) < \mu_o$$

<u>or</u>

$$x(\alpha m - .85K_\epsilon\sqrt{\alpha(1-\alpha)m}) - y(\beta n + .85K_\epsilon\sqrt{\beta(1-\beta)n}) > \mu_o .$$

An approximate lower bound for the significance level of this test is 2γ; an approximate upper bound is 2δ.

Comparison of (5) with (4) shows that for the special case (5), $C_1 = .85K_\epsilon\sqrt{\alpha(1-\alpha)}$, $C_2 = -.85K_\epsilon\sqrt{\beta(1-\beta)}$. Thus the values of C_1 and C_2 chosen for test (5) are not large in magnitude. Similar considerations hold for test (6).

The decision as to when m and n are large enough for these tests to be applicable is a difficult one. Section 6 contains an investigation of this problem for a special case. But there is no reason to believe that the results for this special case hold in most situations of practical interest. In general, however, it seems reasonable to believe that the accuracy of the tests decreases as α and β deviate from $\frac{1}{2}$. For example, a test with $\alpha = .4$, $\beta = .55$ will likely be sufficiently accurate for much smaller values of m and n than a test with $\alpha = .005$, $\beta = .998$. For the sake of definiteness, a rule for deciding when m and n are sufficiently large will be given. It is hoped that this rule will be conservative for most practical situations. In particular, this rule appears to be conservative for the special case of section 6. It should be emphasized, however, that the rule presented is no more than a conjecture and has no strong theoretical or empirical basis. The rule is

<u>Accept that m and n are sufficiently large if</u> $\min[\alpha m, (1-\alpha)m, \beta n, (1-\beta)n] \geq 5$.

In use of this rule, the value of ϵ should not be too small; say, $\epsilon \geq .005$.

4. <u>Derivations</u>. This section contains proof of the results and properties stated in the preceding sections.

First, let us consider the asymptotic distribution of (2). The theorem on which this is based was presented in [5] in a form slightly different from that used in this paper. For completeness and convenience of reference, the version used here will be stated and an outline of the proof presented.

<u>Theorem</u>. <u>Let $z(1), \cdots, z(r)$ denote the values of a sample of size r (arranged in increasing order of magnitude) from a population with probability density function $h(z)$. The function $h(z)$ has the properties that $h(\xi_p) \neq 0$ and that $h'(z)$ exists and is continuous in some neighborhood of ξ_p, where ξ_p is the 100p% point of the population. Then the quantity</u>

$$\sqrt{r/p(1-p)}\, h(\xi_p)\left[z(pr + C\sqrt{r}) - \xi_p\right]$$

<u>has a distribution which approaches the normal distribution with mean</u> $C/\sqrt{p(1-p)}$ <u>and unit variance as</u> $r \to \infty$. <u>Here</u> $pr + C\sqrt{r}$ <u>is restricted to be an integer.</u>

<u>Proof</u>. The method used to prove this theorem is analogous to a modification of that of [1, pp. 368-69] if r is used instead of n and pr is replaced by $pr + C\sqrt{r}$.

The asymptotic distribution of (2) is obtained by applying this theorem to each of the two samples and then considering $x(\alpha m + C_1\sqrt{m}) - y(\beta n + C_2\sqrt{n})$. Explicitly, it is found that the asymptotic distribution of the quantity

$$\frac{x(\alpha m + C_1\sqrt{m}) - y(\beta n + C_2\sqrt{n}) - (\theta_\alpha - \phi_\beta)}{\sqrt{\alpha(1-\alpha)/mf(\theta_\alpha)^2 + \beta(1-\beta)/ng(\phi_\beta)^2}} \tag{7}$$

is normal with unit variance and mean M equal to

$$\left[C_1/\sqrt{m} - C_2 f(\theta_\alpha)/\sqrt{n}\, g(\phi_\beta)\right] \Big/ \sqrt{\alpha(1-\alpha)/m + \beta(1-\beta)f(\theta_\alpha)^2/ng(\phi_\beta)^2} \; .$$

Now consider the choice of C_1 and C_2 so that M is almost constant as a function of $f(\theta_\alpha)$ and $g(\phi_\beta)$. To do this, the value of C_1/C_2 is restricted so that M has the same value for $f(\theta_\alpha)/g(\phi_\beta) = 0$ as for $f(\theta_\alpha)/g(\phi_\beta) = \infty$. This requires that

$$C_1/C_2 = -\sqrt{\alpha(1-\alpha)/\beta(1-\beta)} \; . \tag{8}$$

Using this relation and solving for the maximum and minimum values of M as a function of $f(\theta_\alpha)$ and $g(\phi_\beta)$, it is found that

$$\begin{aligned} &C_1/\sqrt{\alpha(1-\alpha)} < M \le \sqrt{2}\, C_1/\sqrt{\alpha(1-\alpha)} \qquad (C_1 > 0) \\ &\sqrt{2}\, C_1/\sqrt{\alpha(1-\alpha)} \le M < C_1/\sqrt{\alpha(1-\alpha)} \qquad (C_1 < 0). \end{aligned} \tag{9}$$

If $f(\theta_\alpha)/g(\phi_\beta)$ is either very small or very large, the value of M is near $C_1/\sqrt{\alpha(1-\alpha)}$. If $f(\theta_\alpha)/g(\phi_\beta)$ is in the vicinity of 1, the value of M is near $\sqrt{2}\, C_1/\sqrt{\alpha(1-\alpha)}$. Examination of (9) shows that the range of possible variation for M is not great if (8) is satisfied.

Asymptotically, the value of M determines the significance level of a test. For example, let us consider a test of the form (4). Asymptotically, the significance level of this test is determined by

$$(10)\quad \begin{aligned} &\Pr\left[x(\alpha m + C_1\sqrt{m}) - y(\beta n + C_2\sqrt{n}) - (\theta_\alpha - \emptyset_\beta) < 0\right] \\ &= \Pr\left[(7) - M < -M\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-M} e^{-z^2/2}\,dz, \end{aligned}$$

since the asymptotic distribution of (7) - M is standard normal. Thus choice of C_1 and C_2 so that the value of M does not vary much implies that asymptotically the significance level of a test is fairly closely determined without any knowledge of the values of $f(\theta_\alpha)$ and $g(\emptyset_\beta)$.

Comparison with (1) indicates that the distribution of (7) - M is nearly standard normal for values of m and n which are not extremely large, particularly if C_1 and C_2 are not large in magnitude (as is the case for the tests of section 3). Thus it is apparent that the tests of section 3 are usually valid for mediumly large values of m and n. The following intuitive considerations indicate that these tests are applicable even if m and n are not very large.

In (7) and M, let $f(\theta_\alpha)$ and $g(\emptyset_\beta)$ be replaced by the arbitrary positive parameters A and B, respectively. Denote the resulting functions by (7') and M'. For any test, the problem is to fit the cdf of (7') - M' to a certain small part of the cdf of the standard normal distribution by appropriate choice of A and B. Since the part to be fitted is small and the parameters can assume a wide range of values, it seems plausible that this fit can be made rather accurate even for moderately small values of m and n.

For one-sided tests, the interval where the two cdf's are fitted is from the minimum possible value of M' to the maximum possible value of M' (as a function of A and B). The limiting values for M' are given by (9) since M and M' obviously have the same range of possible values. If an accurate fit can be obtained in this interval, the significance level range for moderate m and n is approximately the same as the range for the asymptotic case. As an illustration, consider a test of the form (4). The significance level of this test is approximately determined by the relations

$$(11)\quad \begin{aligned} &\Pr\left[x(\alpha m + C_1\sqrt{m}) - y(\beta n + C_2\sqrt{n}) - (\theta_\alpha - \emptyset_\beta) < 0\right] \\ &= \Pr\left[(7') - M' < -M'\right] \doteq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-M'} e^{-z^2/2}\,dz, \end{aligned}$$

since the cdf of (7') - M' is nearly standard normal in the range of possible values for M'. However, the range of possible values for M' equals the range of possible values for M. The approximate equality of significance level ranges follows from (10).

Now let us consider the determination of the values of C_1 and C_2 used for the tests of section 3. Test (5) was obtained by choosing a representative value of M' in the interval specified by (9) for $C_1 > 0$. Then C_1 was determined so that the significance level of test (4) would be ϵ if M' actually had this value and the cdf of (7') - M' was standard normal. The value of C_2, of course, was found from (8). Using the value of C_1, the relation (11), and the bounds for M', the values of γ and δ could be determined from the relations

$$K_\gamma \doteq 1.175K_\epsilon , \qquad K_\delta \doteq .833K_\epsilon .$$

However, to allow for the fact that the fit to the standard normal cdf in the specified section is only approximate and that A/B is ordinarily not large or near zero, so that M' is usually near $\sqrt{2}\, C_1/\sqrt{\alpha(1-\alpha)}$, the relations used were

$$K_\gamma = 1.25K_\epsilon , \qquad K_\delta = .83K_\epsilon .$$

Similar considerations apply for test (6).

5. <u>Efficiency investigation</u>. Only asymptotic situations will be considered (i.e., both m, $n \to \infty$). It is desired to determine how tests based on (5) and (6) compare with the corresponding tests based on the non-central t-statistic for the case where both populations are normal and the ratio of variances is known.

Let σ_x^2 be the variance of the population with density f(x) while σ_y^2 is the variance of the population with density g(y). Asymptotically, use of the non-central t-statistic for testing $\theta_\alpha - \phi_\beta$ is equivalent to use of the quantity

$$(12) \qquad \left[\bar{x} - \bar{y} - K_\alpha s_x + K_\beta s_y - (\theta_\alpha - \phi_\beta)\right] \Big/ \sqrt{(1 + K_\alpha^2/2)\sigma_x^2/m + (1 + K_\beta^2/2)\sigma_y^2/n} ,$$

where

$$\bar{x} = \sum_1^m x(i)/m, \qquad \bar{y} = \sum_1^n y(j)/n,$$

$$s_x = \sqrt{\sum_1^m \left[x(i) - \bar{x}\right]^2/(m-1)} , \qquad s_y = \sqrt{\sum_1^n \left[y(j) - \bar{y}\right]^2/(n-1)} .$$

The asymptotic distribution of (12) is standard normal.

Since the asymptotic case is being considered, (7) - M is the quantity of interest for tests based on (5) and (6). Comparison of (7) - M and (12) shows that if the sample sizes for (12) were decreased in the ratio

$$(13)\qquad \left[1 + K_\alpha^2/2 + \Phi(1 + K_\beta^2/2)\right] \Big/ 2\pi\left[\alpha(1-\alpha)\exp(K_\alpha^2) + \Phi\beta(1-\beta)\exp(K_\beta^2)\right],$$

where $\Phi = m\sigma_y^2/n\sigma_x^2$, then the asymptotic distributions of (7) - M and (12) would be the same and both (7) - M and (12) would have the same denominator. Thus confidence intervals for $\theta_\alpha - \phi_\beta$ obtained on the basis of (7) - M are asymptotically equivalent to the corresponding confidence intervals obtained on the basis of (12) with m and n decreased in the ratio (13). This property also applies to the significance tests based on these confidence intervals. The ratio (13) will be called the <u>asymptotic efficiency</u> of the tests based on (5) and (6). Actually, (13) is merely the variance of $\bar{x} - \bar{y} - K_\alpha s_x + K_\beta s_y$ divided by the variance of $x(\alpha m + C_1\sqrt{m}) - y(\beta n + C_2\sqrt{n})$ for large m and n.

The value of Φ can lie anywhere between 0 and ∞. Consequently the asymptotic efficiency of the tests based on (5) and (6) can be anywhere between the value of

$$(1 + K_\alpha^2/2) / 2\pi\alpha(1-\alpha)\exp(K_\alpha^2)$$

and the value of

$$(1 + K_\beta^2/2) / 2\pi\beta(1-\beta)\exp(K_\beta^2).$$

Table 1 contains values of

$$(14)\qquad (1 + K_p^2/2) / 2\pi p(1-p)\exp(K_p^2)$$

for p = .01, .02, .05, .10, .20, .30, .40, .50. The value of (14) is the same for p = 1 - v as for p = v. Examination of Table 1 shows that the amount of "information" lost by using tests based on (5) and (6) rather than the non-central t-statistic is not too great if α and β are not near 0 or 1.

6. <u>Investigation of special case</u>. In section 3 a rule was presented for deciding when m and n are sufficiently large for the tests of section 3 to be applicable. The purpose of this section is to check the accuracy of this rule for the special case where $f(x) \equiv g(x)$ and $\alpha = \beta$ (then $\theta_\alpha = \phi_\beta$).

First let us consider some implications of the rule stated in section 3. An obvious result is that $m \geq 10$, $n \geq 10$. Since $.005 \leq \epsilon < .5$, it follows that $0 < K_\epsilon \leq 2.58$. Combining these properties it is seen that the order statistics x(1), x(m), y(1), y(n) are never used for a test when the rule is applied.

Let $m \geq 10$, $n \geq 10$, $n \leq m$ (no loss of generality), and consider the value of $\Pr[x(u) - y(v) < 0]$ for the special case. Here u and v are integers such that $2 \leq u \leq m-1$ and $2 \leq v \leq n-1$. From [6], the value of $\Pr[x(u) - y(v) < 0]$ equals

$$(15) \qquad \frac{\sum_{i=0}^{v-1} \binom{u + v - 2 - i}{u - 1}\binom{m + n - u - v + 1 - i}{m - u}}{\binom{m + n}{m}} .$$

The first step of the empirical analysis consists in selecting suitable values for m, n, α, ϵ. Corresponding values of u and v are then determined according to the requirements for test (5); i.e., u is the integer whose value is nearest to $\alpha m + .85K_\epsilon\sqrt{\alpha(1 - \alpha)m}$, v is the integer whose value is nearest to $\alpha n - .85K_\epsilon\sqrt{\alpha(1 - \alpha)n}$, and both $\alpha m + .85K_\epsilon\sqrt{\alpha(1 - \alpha)m}$ and $\alpha n - .85K_\epsilon\sqrt{\alpha(1 - \alpha)n}$ are nearly equal to integers. Then the value of (15) is computed and compared with the corresponding values of γ and δ. This is done for $m \geq n = 10, 15, 20, 25$ and various values of α and ϵ. The results of these computations are contained in Table 2.

Examination of Table 2 shows that the value of (15) is usually near γ. This is to be expected for the special situation considered. The derivation of γ and δ shows that δ is approached when A/B is near 0 or ∞ while γ is approximated if A/B is in the vicinity of 1. Since the populations are the same for the case considered, the value of A/B should not differ greatly from unity. Thus the values obtained for (15) can be considered very good approximations to those expected for situations of this type. Consequently the rule of section 3 would seem to be adequate for this special case (i.e., $f \equiv g$, $\alpha = \beta$).

REFERENCES

[1] Harald Cramér, Mathematical Methods of Statistics, Princeton Univ. Press, 1946.

[2] Frederick Mosteller, "On some useful 'inefficient' statistics," Annals of Math. Stat., Vol. 17 (1946), pp. 377-408.

[3] Herman Chernoff, "Asymptotic Studentization in testing of hypotheses," Annals of Math. Stat., Vol. 20 (1949), pp. 268-78.

[4] John E. Walsh, "Some large sample tests for the median." Abstracted in Annals of Math. Stat., Vol. 20 (1949), pp. 468-69.

[5] John E. Walsh, "Some estimates and tests based on the r smallest values in a sample." To appear in Annals of Math. Stat.

[6] William R. Thompson, "Biological application of normal range and associated significance tests in ignorance of original distribution forms," Annals of Math. Stat., Vol. 9 (1938), pp. 281-87.

TABLE 1

Values of $(1 + K_p^2/2) \,/\, 2\pi p(1 - p)\exp(K_p^2)$

p	.01	.02	.05	.10	.20	.30	.40	.50
(14)	.265	.370	.526	.625	.662	.654	.645	.637

TABLE 2

Numerical Comparison of (15) with γ and δ for Special Case

n	m	α	ϵ	u	v	(15)	γ	δ
10	10	.45	.030	7	2	.010	.009	.059
		.50	.013	8	2	.003	.003	.032
		.55	.030	8	3	.012	.009	.059
	15	.46	.025	10	2	.007	.007	.052
		.50	.014	11	2	.003	.003	.034
		.54	.036	11	3	.013	.012	.068
	20	.405	.051	11	2	.021	.020	.087
		.465	.023	13	2	.006	.006	.049
		.535	.038	14	3	.013	.013	.070
		.595	.012	16	3	.003	.002	.030
	25	.42	.045	14	2	.015	.017	.080
		.445	.031	15	2	.009	.010	.061
		.555	.025	18	3	.007	.007	.052
		.58	.016	19	3	.004	.004	.038
15	15	.30	.048	7	2	.018	.019	.084
		.40	.030	9	3	.011	.009	.059
		.50	.016	11	4	.005	.004	.038
		.60	.030	12	6	.013	.009	.059
		.70	.048	13	8	.025	.019	.083
	20	.39	.040	11	3	.013	.014	.073
		.50	.060	13	5	.028	.026	.098
		.61	.022	16	6	.007	.006	.047
	25	.38	.044	13	3	.015	.016	.078
		.455	.042	15	4	.015	.015	.076
		.545	.023	18	5	.007	.006	.049
		.62	.017	20	6	.004	.004	.039
20	20	.25	.033	8	2	.010	.011	.064
		.40	.050	11	5	.024	.020	.086
		.60	.050	15	9	.027	.020	.086
		.75	.033	18	12	.016	.011	.064
	25	.305	.037	11	3	.013	.013	.069
		.50	.051	16	7	.024	.021	.087
		.705	.036	21	11	.017	.012	.068
25	25	.20	.038	8	2	.012	.013	.070
		.40	.026	14	6	.009	.008	.053
		.60	.026	19	11	.010	.008	.053
		.80	.038	23	17	.019	.013	.070